Izvod f-je

Definicija Neka je f-ja f definisana na otvorenom intervalu (a,b) i neka je $c\in(a,b)$. Kažemo da f ima izvod (ili derivaciju) u tački c ako postoji limes $\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}$. Vrijednost limesa obilježavamo sa $f'(c)$ i zovemo izvod f-je f u tački c.

1. Korištenjem navedene definicije naći izvode u tački c sljedećih f-ja:

a) $y=x$ c) $y=\cos x$ e) $y=x^2$

b) $y=\sqrt[3]{x}$ d) $y=x^\alpha,\ \alpha\in\mathbb{R}$ f) $y=\sin x$

Rj. a) $f(x)=x$, $f'(c)=\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim\limits_{x\to c}\frac{x-c}{x-c}=\lim\limits_{x\to c}1=1$

$$\Rightarrow\quad (x)'=1$$

b) $f(x)=\sqrt[3]{x}$, $f'(c)=\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim\limits_{x\to c}\frac{\sqrt[3]{x}-\sqrt[3]{c}}{x-c}\cdot\frac{(\sqrt[3]{x^2}+\sqrt[3]{xc}+\sqrt[3]{c^2})}{(\sqrt[3]{x^2}+\sqrt[3]{xc}+\sqrt[3]{c^2})}$

$$=\lim_{x\to c}\frac{\cancel{x-c}}{\cancel{(x-c)}(\sqrt[3]{x^2}+\sqrt[3]{xc}+\sqrt[3]{c^2})}=\frac{1}{3\sqrt[3]{c^2}}\quad\Rightarrow\quad(\sqrt[3]{x})'=\frac{1}{3\sqrt[3]{x^2}}$$

c) $f(x)=\cos x$, $f'(c)=\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim\limits_{x\to c}\frac{\cos x-\cos c}{x-c}\overset{(*)}{=}$

$$\cos x=\cos\frac{x+c+x-c}{2}=\cos\left(\frac{x+c}{2}+\frac{x-c}{2}\right)=\cos\frac{x+c}{2}\cos\frac{x-c}{2}-\sin\frac{x+c}{2}\sin\frac{x-c}{2}$$

$$\cos c=\cos\frac{x+c-x+c}{2}=\cos\left(\frac{x+c}{2}-\frac{x-c}{2}\right)=\cos\frac{x+c}{2}\cos\frac{x-c}{2}+\sin\frac{x+c}{2}\sin\frac{x-c}{2}$$

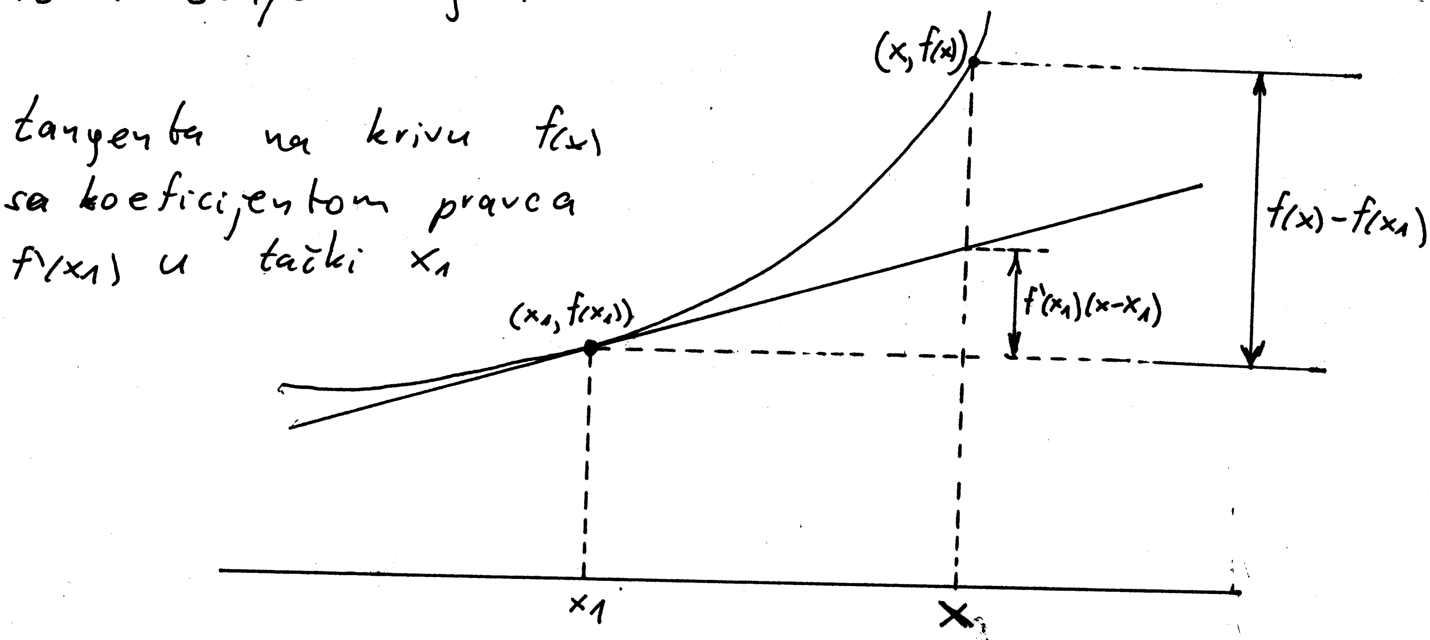
$$\cos x-\cos c=-2\sin\frac{x+c}{2}\sin\frac{x-c}{2}$$

$$\overset{(*)}{=}\lim_{x\to c}\frac{-2\sin\frac{x+c}{2}\sin\frac{x-c}{2}}{x-c}=-\lim_{x\to c}\sin\frac{x+c}{2}\cdot\underbrace{\lim_{x\to c}\frac{\sin\frac{x-c}{2}}{\frac{x-c}{2}}}_{1}=-\sin c\quad\Rightarrow(\cos x)'=-\sin x$$

Ako f-ja $f(x)$ ima izvod u tački c tada je $f(x)$ neprekidna u tački c.

Izvodi se upotrebljavaju u mnogim problemima, a najvažnije dvije skupine su:

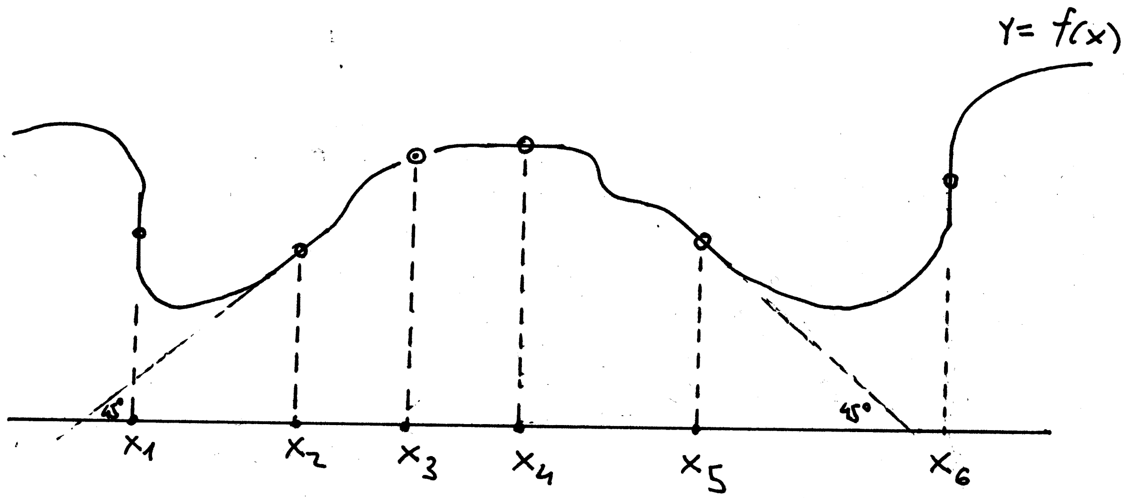
1. određivanje brzine tačke koja se kreće pravoliniski
2. iznalaženje tangente na krivu



$$Y - Y_1 = k(x - x_1)$$

$f(x) - f(x_1) = f'(x_1)(x - x_1)$ jednačina tangente na krivu $y = f(x)$ u nekoj tački $(x_1, f(x_1))$

$k_1 \cdot k_2 = -1$ uslov normalnosti dvije prave



$f'(x_1) = -\infty$ $\quad$ $f'(x_3)$ ne postoji $\quad$ $f'(x_5) = -1$

$f'(x_2) = 1$ $\quad$ $f'(x_4) = 0$ $\quad$ $f'(x_6) = \infty$

Tablica izvoda

1. $c' = 0$, c - konst.
2. $(x^{\alpha})' = \alpha x^{\alpha - 1}$, $\alpha \in \mathbb{R}$

   $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$, $x > 0$
3. $(a^{x})' = a^{x} \ln a$

   $(e^{x})' = e^{x}$
4. $(\log_a x)' = \dfrac{1}{x \ln a}$

   $(\ln x)' = \dfrac{1}{x}$
5. $(\sin x)' = \cos x$
6. $(\cos x)' = -\sin x$
7. $(tg\, x)' = \dfrac{1}{\cos^2 x}$
8. $(ctg\, x)' = -\dfrac{1}{\sin^2 x}$
9. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$, $|x| < 1$
10. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$, $|x| < 1$
11. $(arc\,tg\, x)' = \dfrac{1}{1+x^2}$
12. $(arc\,ctg\, x)' = -\dfrac{1}{1+x^2}$

$$\boxed{sh\, x = \frac{e^{x} - e^{-x}}{2} \qquad ch\, x = \frac{e^{x} + e^{-x}}{2}}$$

13. $(sh\, x)' = ch\, x$
14. $(ch\, x)' = sh\, x$
15. $(th\, x)' = \dfrac{1}{ch^2 x}$
16. $(cth\, x)' = -\dfrac{1}{sh^2 x}$

Pravila izvoda:

1. $(f \pm g)'(c) = f'(c) \pm g'(c)$
2. $(f \cdot g)'(c) = f'(c)\, g(c) + f(c)\, g'(c)$
3. $(\alpha f)'(c) = \alpha f'(c)$
4. $\left(\dfrac{f}{g}\right)'(c) = \dfrac{f'(c)\, g(c) - f(c)\, g'(c)}{g(c)^2}$, $g(c) \neq 0$

1. Izračunati izvode f-ja:

a) $y = x^5 - 4x^3 + 2x - 3$

Rj. $y' = 5x^4 - 12x^2 + 2$

b) $y = ax^2 + bx + c$

Rj. $y = 2ax + b$

c) $y = -\frac{5x^3}{a}$

Rj. $y' = -\frac{5}{a}(x^3)' = -\frac{15}{a}x^2$

d) $y = x^2 \sqrt[3]{x^2}$

Rj. $y = x^2 \cdot x^{\frac{2}{3}} = x^{\frac{8}{3}}$

$y' = \frac{8}{3} x^{\frac{5}{3}} = \frac{8}{3}\sqrt[3]{x^5} = \frac{8}{3} x \sqrt[3]{x^2}$

e) $y = \frac{a+bx}{c+dx}$

Rj. $y' = \frac{b(c+dx) - (a+bx)\cdot d}{(c+dx)^2}$

$y' = \frac{bc + bdx - ad - bdx}{(c+dx)^2}$

$y' = \frac{bc - ad}{(c+dx)^2}$

f) $y = \frac{2}{2x-1} - \frac{1}{x}$, znamo: $\frac{1}{x} = x^{-1}$

Rj. $y' = \frac{0(2x-1) - 2(2)}{(2x-1)^2} - (-1)x^{-2}$

$y' = \frac{-4}{(2x-1)^2} + \frac{1}{x^2}$

$y' = \frac{1-4x}{x^2(2x-1)^2}$

g) $y = \frac{ax^6 + b}{\sqrt{a^2+b^2}}$

Rj. $y = \frac{a}{\sqrt{a^2+b^2}} x^6 + \frac{b}{\sqrt{a^2+b^2}}$

$y' = \frac{6a}{\sqrt{a^2+b^2}} x^5$

h) $y = 3x^{\frac{2}{3}} - 2x^{\frac{5}{2}} + x^{-3}$

Rj. $y' = 3\cdot\frac{2}{3}x^{-\frac{1}{3}} - 2\cdot\frac{5}{2}x^{\frac{3}{2}} - 3x^{-4}$

$= 2x^{-\frac{1}{3}} - 5x^{\frac{3}{2}} - 3x^{-4}$

i) $y = \frac{2x+3}{x^2-5x+5}$

Rj. $y' = \frac{2(x^2-5x+5) - \overbrace{(2x+3)(2x-5)}^{4x^2-10x+6x-15}}{(x^2-5x+5)^2}$

$y' = \frac{2x^2 - 10x + 10 - 4x^2 + 4x + 15}{(x^2-5x+5)^2}$

$y' = \frac{-2x^2 - 6x + 25}{(x^2-5x+5)^2}$

2. Izračunati izvode f-ja:

a) $y = at^m + bt^{m+n}$ Rj. $y' = mat^{m-1} + b(m+n)t^{m+n-1}$

b) $y = \dfrac{a}{\sqrt[3]{x^2}} - \dfrac{b}{x\sqrt[3]{x}}$, Rj. $y' = \dfrac{4b}{3x^2\sqrt[3]{x}} - \dfrac{2a}{3x\sqrt[3]{x^2}}$

c) $y = \dfrac{1+\sqrt{z}}{1-\sqrt{z}}$, $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$

Rj.
$$y' = \frac{\frac{1}{2\sqrt{z}}(1-\sqrt{z}) - (1+\sqrt{z})\left(-\frac{1}{2\sqrt{z}}\right)}{(1-\sqrt{z})^2} = \frac{\frac{1-\cancel{\sqrt{z}}+1+\cancel{\sqrt{z}}}{2\sqrt{z}}}{(1-\sqrt{z})^2} = \frac{1}{(1-\sqrt{z})^2\sqrt{z}}$$

d) $y = \operatorname{tg} x - \operatorname{ctg} x$

Rj.
$$y' = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x} = \frac{1}{\sin^2 x\cos^2 x} = \frac{4}{(2\sin x\cos x)^2}$$

$$y' = \frac{4}{\sin^2 2x}$$

e) $y = \dfrac{\pi}{x} + \ln 2$, Rj. $y' = -\dfrac{\pi}{x^2}$

f) $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

Rj.
$$y' = \frac{(\cos x - \sin x)(\sin x - \cos x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$y' = \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} = \frac{-(\sin^2 x - \cancel{2\sin x\cos x} + \cos^2 x + \sin^2 x + \cancel{2\sin x\cos x} + \cos^2 x)}{(\sin x - \cos x)^2}$$

$$y' = \frac{-2}{(\sin x - \cos x)^2}$$

g) $y = 2t\sin t - (t^2-2)\cos t$

Rj.
$$y' = 2(1\cdot\sin t + t\cos t) - [2t\cos t + (t^2-2)(-\sin t)] =$$
$$= 2\sin t + \cancel{2t\cos t} - \cancel{2t\cos t} + (t^2-2)\sin t = 2\sin t + (t^2-2)\sin t =$$
$$= 2\sin t + t^2\sin t - 2\sin t$$
$$y' = t^2\sin t$$

$y = x\arcsin x$

Rj. $y' = \arcsin x + x\cdot\frac{1}{\sqrt{1-x^2}}$

$y=(x-1)e^x$

Rj. $y' = e^x + (x-1)e^x$

$y' = e^x(1+x-1) = xe^x$

$\boxed{\log_B A = \frac{\ln A}{\ln B}}$

$y = \frac{x^5}{e^x}$

Rj. $y' = \frac{5x^4e^x - x^5e^x}{e^{2x}} = \frac{x^4e^x(5-x)}{(e^x)^2}$

$y' = \frac{x^4(5-x)}{e^x}$

$y = \frac{x^2}{\ln x}$

Rj. $y' = \frac{2x\cdot\ln x - x^2\cdot\frac{1}{x}}{\ln^2 x} = \frac{2x\ln x - x}{\ln^2 x}$

$y' = \frac{x(2\ln x - 1)}{\ln^2 x}$

$y = \ln x\,\underbrace{\lg x}_{=\log_{10} x} - \ln a \cdot \log_a x$

Rj. $y' = \frac{1}{x}\log_{10} x + \frac{\ln x}{x\ln 10} - \cancel{\ln a}\,\frac{1}{x\cancel{\ln a}}$

$y' = \frac{1}{x}\frac{\ln x}{\ln 10} + \frac{\ln x}{x\ln 10} - \frac{1}{x}$

$y' = \frac{2\ln x}{x\ln 10} - \frac{1}{x}$

$y = x\,\mathrm{ctg}\,x$ Rj. $y' = \mathrm{ctg}\,x - \frac{x}{\sin^2 x}$

$y = \frac{(1+x^2)\,\mathrm{arctg}\,x - x}{2}$ Rj. $y' = x\,\mathrm{arctg}\,x$

$y = \frac{1}{x} + 2\ln x - \frac{\ln x}{x}$ Rj. $y' = \frac{2}{x} + \frac{\ln x}{x^2} - \frac{2}{x^2}$

$\log_B A = \frac{\log_a A}{\log_a B}$, $\ln x = \log_e x$, $\log_a B = \frac{1}{\log_B a}$

Izvodi složenih f-ja

$y = f(g(x))$, $y'_x = f'_s \cdot g'_x$ ili $\left.\begin{array}{l} y = \psi(u) \\ u = \varphi(x) \end{array}\right\}$ $y = \psi(\varphi(x))$

$y'_x = y'_u \cdot u'_x$

1. Naći izvode sljedećih f-ja:

a) $y = (1+3x-5x^2)^{30}$

Rj. $y = u^{30}$, gdje je $u = 1+3x-5x^2$

$y' = 30u^{29} \cdot u'$, $u' = 3-10x$

$y' = 30(1+3x-5x^2)^{30} \cdot (3-10x)$

b) $y = (3+2x^2)^4$

Rj. $y' = 4(3+2x^2)^3 \cdot (3+2x^2)'$

$y' = 4(3+2x^2)^3 \cdot 4x = 16x(3+2x^2)^3$

c) $y = \sqrt[3]{a+bx^3}$

Rj. $y = \sqrt[3]{u}$, $u = a+bx^3$

$y' = \frac{1}{3}u^{-\frac{2}{3}} \cdot u'$, $u' = 3bx^2$

$y' = \frac{1}{\cancel{3}u^{\frac{2}{3}}} \cdot \cancel{3}bx^2$

$y' = \frac{bx^2}{\sqrt[3]{(a+bx^3)^2}}$

d) $f(y) = (2a+3by)^2$

Rj. $f'(y) = 12ab + 18b^2y$

e) $y = \sqrt{ctg\,x} - \sqrt{ctg\,\alpha}$

Rj. $y = \sqrt{u} - \sqrt{ctg\,\alpha}$, $u = ctg\,x$

$y' = \frac{1}{2\sqrt{u}} \cdot u'$, $u' = -\frac{1}{\sin^2 x}$

$y' = \frac{-1}{2\sin^2 x\sqrt{ctg\,x}}$

f) $y = 2x + 5\cos^3 x$

Rj. $y' = 2 + 15\cos^2 x \cdot (-\sin x)$

$y' = 2 - 15\cos^2 x \sin x$

g) $f(x) = -\frac{1}{6(1-3\cos x)^2}$

Rj. $y' = \frac{\sin x}{(1-3\cos x)^3}$

Naći izvode sljedećih f-ja:

$y = x^4(a-2x^3)^2$

Rj. $y' = 4x^3(a-2x^3)^2 + x^4\cdot 2(a-2x^3)\overbrace{\cdot(-6)}^{(-2)(3)}x^2$

$y' = 4x^3(a-2x^3)\cdot\left[\underbrace{a-2x^3+x\cdot(-1)\cdot 3x^2}_{a-2x^3-3x^3}\right]$

$y' = 4x^3(a-2x^3)(a-5x^3)$

$y = (a+x)\sqrt{a-x}$

Rj. $y' = 1\cdot\sqrt{a-x} + (a+x)\dfrac{1}{2\sqrt{a-x}}\cdot(-1)$

$y' = \sqrt{a-x} - \dfrac{a+x}{2\sqrt{a-x}} = \dfrac{2(a-x)-(a+x)}{2\sqrt{a-x}}$

$y' = \dfrac{a-3x}{2\sqrt{a-x}}$

$z = \sqrt[3]{y+\sqrt{y}}$

Rj. $(\sqrt[3]{x})' = (x^{\frac{1}{3}})' = \frac{1}{3}x^{-\frac{2}{3}} = \dfrac{1}{3\sqrt[3]{x^2}}$

$z' = \dfrac{1}{3\sqrt[3]{(y+\sqrt{y})^2}}\cdot(y+\sqrt{y})'$

$z' = \dfrac{1}{3\sqrt[3]{(y+\sqrt{y})^2}}\cdot\left(1+\dfrac{1}{2\sqrt{y}}\right)$

$z' = \dfrac{1}{3\sqrt[3]{(y+\sqrt{y})^2}}\cdot\dfrac{2\sqrt{y}+1}{2\sqrt{y}}$

$z' = \dfrac{2\sqrt{y}+1}{6\sqrt{y}\sqrt[3]{(y+\sqrt{y})^2}}$

$y = tg^2 5x$

Rj. $y' = 2\,tg5x\cdot(tg5x)'$

$y' = 2\,tg5x\cdot\dfrac{1}{\cos^2 x}\cdot(5x)'$

$y' = \dfrac{10\,tg5x}{\cos^2 x}$

$y = \sqrt{\cos x}\cdot a^{\sqrt{\cos x}}$

Rj. $y' = \dfrac{1}{2\sqrt{\cos x}}\cdot(\cos x)'\cdot a^{\sqrt{\cos x}} + \sqrt{\cos x}\cdot a^{\sqrt{\cos x}}\ln a\cdot(\sqrt{\cos x})'$

$y' = -\dfrac{\sin x}{2\sqrt{\cos x}}\cdot a^{\sqrt{\cos x}} + \ln a\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}\cdot\dfrac{1}{2\sqrt{\cos x}}\cdot(\cos x)'$

$y' = -\dfrac{\sin x}{2\sqrt{\cos x}}a^{\sqrt{\cos x}} - \dfrac{\ln a\cdot\sin x\cdot\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}}{2\sqrt{\cos x}}$

$y' = -\dfrac{\sin x\, a^{\sqrt{\cos x}}}{2\sqrt{\cos x}}\left[1+\ln a\cdot\sqrt{\cos x}\right]$

$y' = -\dfrac{\sin x\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}}{2\sqrt{\cos x}\cdot\sqrt{\cos x}}\left[1+\ln a\cdot\sqrt{\cos x}\right]$

$y' = -\frac{1}{2}tgx\cdot y\cdot\left[1+\ln a\sqrt{\cos x}\right]$

$y = 3^{ctg\frac{1}{x}}$ Rj. $y' = \dfrac{3^{ctg\frac{1}{x}}\cdot\ln 3}{(x\sin\frac{1}{x})^2}$

$y = \ln(x+\sqrt{a^2+x^2})$ Rj. $y' = \dfrac{1}{\sqrt{a^2+x^2}}$

(#) $y=\ln\frac{(x-2)^5}{(x+1)^3}$

Rj.
$$y=\ln(x-2)^5-\ln(x+1)^3$$
$$y'=\frac{1}{(x-2)^5}\cdot((x-2)^5)'-\frac{1}{(x+1)^3}\cdot[(x+1)^3]'$$
$$y'=\frac{5(x-2)^4}{(x-2)^5}-\frac{3(x+1)^2}{(x+1)^3}\ \ldots$$

y mogu napisati i kao
$$y=5\ln(x-2)-3\ln(x+1)$$
$$y'=5\cdot\frac{1}{x-2}-3\cdot\frac{1}{x+1}$$
$$y'=\frac{5(x+1)-3(x-2)}{(x-2)(x+1)}$$
$$y'=\frac{2x+11}{x^2-x-2}$$

(#) $y=\ln\ln(3-2x^3)$

Rj. $$y'=\frac{1}{\ln(3-2x^3)}\cdot(\ln(3-2x^3))'$$
$$y'=\frac{1}{\ln(3-2x^3)}\cdot\frac{1}{3-2x^3}\cdot(3-2x^3)'$$
$$y'=\frac{-6x^2}{(3-2x^3)\ln(3-2x^3)}$$

(#) $y=\ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}$

Rj. prvo pojednostavimo izraz
$$\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}\cdot\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}+x}=$$
$$=\frac{(\sqrt{x^2+a^2}+x)^2}{x^2+a^2-x^2}=\frac{(\sqrt{x^2+a^2}+x)^2}{a^2}$$
$$y=\ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}=2\ln\frac{\sqrt{x^2+a^2}+x}{a^2}$$
$$y'=2\cdot\frac{1}{\frac{\sqrt{x^2+a^2}+x}{a^2}}\cdot\left(\frac{\sqrt{x^2+a^2}+x}{a^2}\right)'$$
$$y'=\frac{2\cancel{a^2}}{\sqrt{x^2+a^2}+x}\cdot\frac{1}{\cancel{a^2}}\cdot\left[\frac{1}{\cancel{2}\sqrt{x^2+a^2}}\cdot\overbrace{(x^2+a^2)'}^{\cancel{2}x}+1\right]$$
$$y'=\frac{2}{\sqrt{x^2+a^2}+x}\cdot\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$$
$$y'=\frac{2}{\sqrt{x^2+a^2}}$$

(#) $y=\operatorname{arctg}\ln x$

Rj. $$y'=\frac{1}{1+\ln^2x}\cdot(\ln x)'$$
$$y'=\frac{1}{x(1+\ln^2x)}$$

(#)ˇ $y=\ln\frac{(x-1)^3(x-2)}{x-3}$ Rj. $y'=\frac{3x^2-16x+19}{(x-1)(x-2)(x-3)}$

(#)ˇ $f(x)=\sqrt{x^2+1}-\ln\frac{1+\sqrt{x^2+1}}{x}$ Rj. $y'=\frac{\sqrt{1+x^2}}{x}$

Izvodi f-ja koje nisu eksplicitno zadane

$y=f(x)$ je eksplicitni oblik f-je. Pored eksplicitnog oblika postoje: $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$ parametarski oblik f-je

i $F(x,y)=0$ implicitan oblik f-je

1. Izračunati $y'=\frac{dy}{dx}$ ako je f-ja y zadana parametarski $\begin{cases} x=a\cos t \\ y=a\sin t \end{cases}$.

Rj. $\frac{dx}{dt}=-a\sin t \qquad \frac{dy}{dt}=a\cos t$

$$\frac{dy}{dx}=\frac{a\cos t}{-a\sin t}=-\operatorname{ctg} t$$

tj. $y'=-\operatorname{ctg} t$

2. Izračunati $y'=\frac{dy}{dx}$ ako je f-ja y zadana $\begin{cases} x=\sqrt{t} \\ y=\sqrt[3]{t} \end{cases}$.

Rj. $\frac{dx}{dt}=\frac{1}{2\sqrt{t}}, \quad \frac{dy}{dt}=\frac{1}{3}t^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{t^2}}$

$$\frac{dy}{dx}=\frac{\frac{1}{3\sqrt[3]{t^2}}}{\frac{1}{2\sqrt{t}}}=\frac{2\sqrt{t}}{3\sqrt[3]{t^2}}=\frac{2}{3}\sqrt[6]{\frac{t^3}{t^4}}=\frac{2}{3\sqrt[6]{t}}$$

tj. $y'=\frac{2}{3\sqrt[6]{t}}$

3. Izračunati $y'=\frac{dy}{dx}$ ako je f-ja y zadana par. $\begin{cases} x=a\cos^3 t \\ y=b\sin^3 t \end{cases}$

Rj. $y'=-\frac{b}{a}\operatorname{tg} t$

4. Izračunati izvod y'_x ako je f-ja zadana implic. $x^3+y^3-3axy=0$.

Rj. $x^3+y^3-3axy=0$

$3x^2+\underline{3y^2\cdot y'}-3ay\underline{-3axy'}=0$

$(3y^2-3ax)y'=3ay-3x^2 \quad /:3$

$$y'=\frac{ay-x^2}{y^2-ax}$$

5. Izračunati izvod y'_x ako je f-ja zadana implicitno $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Rj. $\frac{1}{a^2}\cdot 2x+\frac{1}{b^2}\cdot 2y\cdot y'=0$

$\frac{2y}{b^2}y'=-\frac{2x}{a^2} \quad /:2$

$$y'=-\frac{xb^2}{ya^2}$$

6. Izračunati izvod y'_x ako je f-ja zadana implicitno $\sqrt{x^2+y^2}=c\cdot\operatorname{arctg}\frac{y}{x}$.

Rj. $y'=\frac{cy+x\sqrt{x^2+y^2}}{cx-y\sqrt{x^2+y^2}}$

Logaritamski izvod

Logaritamskim izvodom f-je $y=f(x)$ nazivamo izvodom logaritma te f-je tj. $(\ln y)' = \frac{y'}{y} = \frac{f'(x)}{f(x)}$.

(1.) Naći izvod složene eksplicitno zadane f-je $y=u^v$ ako je $u=\varphi(x)$ i $v=\psi(x)$.

Rj. $y=u^v \quad /\ln$

$\ln y = \ln u^v$

$\ln y = v\ln u \quad /'$

$\frac{1}{y}\cdot y' = v'\ln u + v\cdot\frac{1}{u}\cdot u' \quad /\cdot y$

$y' = y\left(v'\ln u + \frac{v}{u}u'\right)$

(2.) Izračunati y' ako je $y=(\sin x)^x$.

Rj. $y=(\sin x)^x \quad /\ln$

$\ln y = \ln(\sin x)^x$

$\ln y = x\ln\sin x \quad /'$

$\frac{1}{y}\cdot y' = \ln\sin x + x\frac{1}{\sin x}\cdot\overbrace{(\sin x)'}^{\cos x}$

$y' = y\left(\ln\sin x + x\cdot\frac{\cos x}{\sin x}\right)$

$y' = (\sin x)^x\left(\ln\sin x + x\,\mathrm{ctg}\,x\right)$

(3.) Izračunati y' ako je $y=\sqrt[3]{x^2}\cdot\frac{1-x}{1+x^2}\cdot\sin^3 x\cdot\cos^2 x$.

Rj. $\ln y = \ln\sqrt[3]{x^2} + \ln\frac{1-x}{1+x^2} + \ln\sin^3 x + \ln\cos^2 x$

$\ln y = \frac{2}{3}\ln x + \ln\frac{1-x}{1+x^2} + \ln\sin^3 x + \ln\cos^2 x \quad /'$

$$\left(\frac{1-x}{1+x^2}\right)' = \frac{(-1)(1+x^2)-(1-x)\cdot 2x}{(1+x^2)^2} = \frac{-1-x^2-2x+2x^2}{(1+x^2)^2} = \frac{x^2-2x-1}{(1+x^2)^2}$$

$\frac{1}{y}\cdot y' = \frac{2}{3}\cdot\frac{1}{x} + \frac{\cancel{1+x^2}}{1-x}\cdot\frac{x^2-2x-1}{(1+x^2)^{\cancel{2}}} + \frac{3\cancel{\sin^2 x}}{\sin^{\cancel{3}} x}\cdot\cos x + \frac{2\cancel{\cos x}}{\cos^{\cancel{2}} x}\cdot(-\sin x)$

$y' = y\left(\frac{2}{3x} + \frac{x^2-2x-1}{(1-x)(1+x^2)} + 3\,\mathrm{ctg}\,x - 2\,\mathrm{tg}\,x\right)$

(4.) $y=x^x$, Rj. $y' = x^x(1+\ln x)$

(5.) $y=x^{x^2}$, Rj. $y' = x^{x^2+1}(1+2\ln x)$

(6.) $y=\sqrt[x]{x}$, Rj. $y' = \sqrt[x]{x}\,\frac{1-\ln x}{x^2}$

Primjena izvoda u geometriji

Ako je data kriva $y=f(x)$ i ako je $M(x_1,y_1)$ data tačka tada je $y-y_1=f'(x_1)(x-x_1)$ jednačina tangente u tački M

$$x-x_1+f'(x_1)(y-y_1)=0 \quad \text{ili} \quad y-y_1=\frac{-1}{f'(x_1)}(x-x_1)$$

je jednačina normale na krivu tački $M(x_1,y_1)$

Ako su $y_1=k_1x+n_1$ i $y_2=k_2x+n_2$ dvije date prave tada je

$tg\varphi=\dfrac{k_2-k_1}{1+k_1\cdot k_2}$ tangens ugla između dvije prave

Pod uglom između dvije krive $y=f_1(x)$ i $y=f_2(x)$ u njihovoj presječnoj tački podrazumjevamo uga φ između njihovih zajednički tangenti u presječnoj tački $N(x_1,y_1)$

$$tg\varphi=\frac{f_2'(x_1)-f_1'(x_1)}{1+f_1'(x_1)\cdot f_2'(x_1)}$$

1. Naći jednačinu tangente na krivu $y=2x^2-4x-6$ u tački $M(\frac{3}{2},\frac{-15}{2})$ i nacrtati sliku.

Rj. $y=2x^2-4x-6$

nacrtajmo ovu krivu

nule $y=0$

$2x^2-4x-6=0$

$2(x^2-2x-3)=0$

$2(x+1)(x-3)=0$

$x_1=3 \Rightarrow y=0$

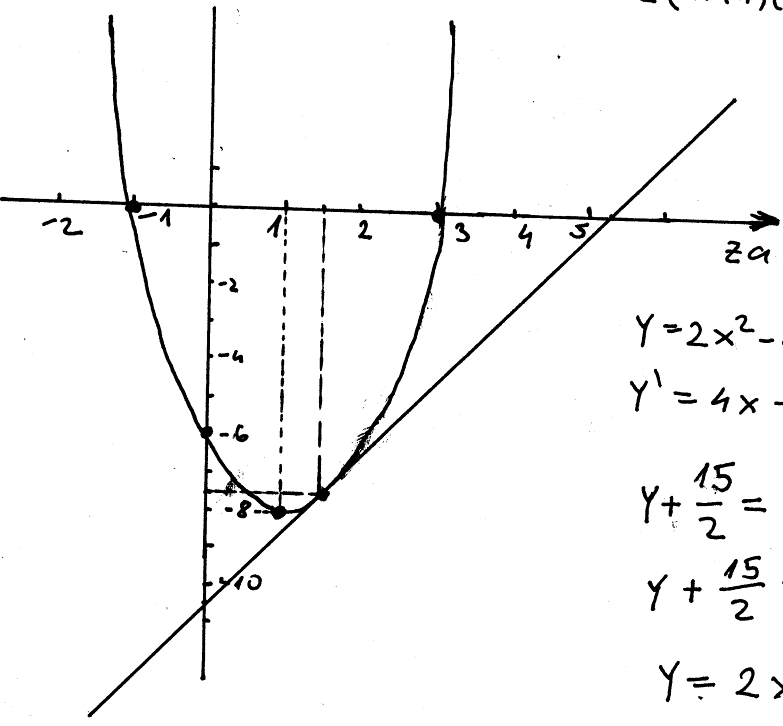
$x_2=-1 \Rightarrow y=0$

tjeme parabole

$$T\left(-\frac{b}{2a},-\frac{D}{4a}\right)$$

$$-\frac{b}{2a}=\frac{4}{4}=1$$

$$-\frac{D}{4a}=-\frac{16+48}{8}=-\frac{64}{8}=-8$$



za $x=0 \Rightarrow y=-6$

$y=2x^2-4x-6$

$y'=4x-4$

$M\in f(x)$

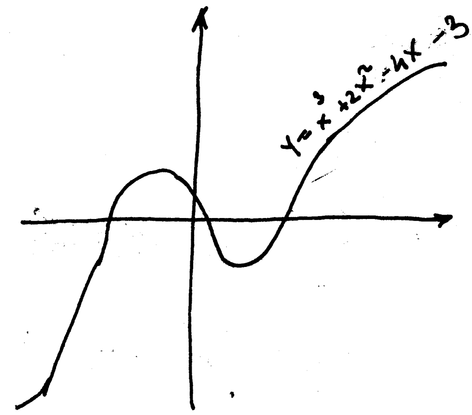
$$y'\left(\frac{3}{2}\right)=4\cdot\frac{3}{2}-4=6-4=2$$

$$y+\frac{15}{2}=2\left(x-\frac{3}{2}\right)$$

$$y+\frac{15}{2}=2x-3$$

$y=2x-\frac{21}{2}$ jednačina tangente

(2.) Napišite jednačinu tangente i normale na krivu $y=x^3+2x^2-4x-3$ u tački $(-2,5)$.



Rj. $y'=3x^2+4x-4$

$y'(-2)=12-8-4=0$

$y-y_0=f'(x_0)(x-x_0)$

$y-5=0(x+2)$

$y-5=0$ jednačina tangente

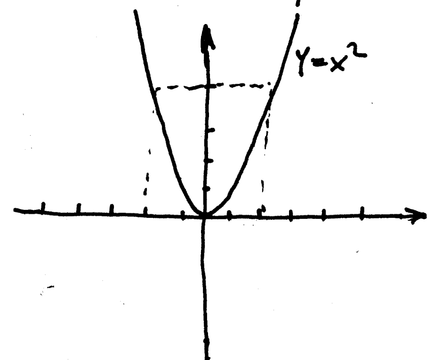
$x-x_0+y_0'(y-y_0)=0$ jedn. norm.

$x+2=0$ jedn. normale

(3.) Naći jednačinu tangente i normale na krivu $y=\sqrt[3]{x-1}$ u tački $(1,0)$. Rj. $x-1=0$, $y=0$

(4.) Odrediti ugao pod kojim se sijeku krive $y=x^2$ i $x=y^2$!

Rj. Prvo nađimo tačke presjeka krivih.



$$\begin{aligned} y&=x^2 \\ x&=y^2 \\ \hline y&=y^4 \end{aligned}$$

$y-y^4=0$

$y^4-y=0$

$y(y^3-1)=0$

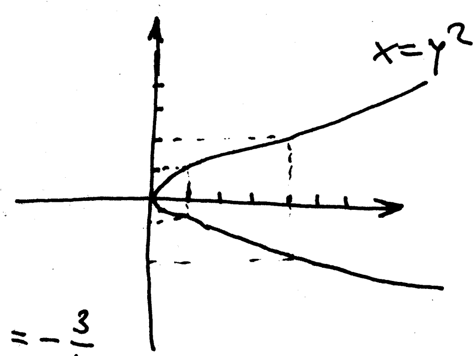
$y(y-1)(y^2+y+1)=0$

$y_1=0$ ili $y_2=1$

$y_1=0 \Rightarrow x_1=0$

$y_2=1 \Rightarrow x_2=1$

Postoje dvije tačke presjeka $(0,0)$ i $(1,1)$



f_1: $y=x^2$

$y'=2x$

$f_1'(0)=0$

$f_1'(1)=2$

f_2: $x=y^2$

$1=2yy'$

$y'=\frac{1}{2y}$

$f_2'(0)$ nije def.

$f_2'(1)=\frac{1}{2}$

$$tg\varphi=\frac{f_2'(x_0)-f_1'(x_0)}{1-f_1'(x_0)\cdot f_2'(x_0)}$$

$$tg\varphi=\frac{\frac{1}{2}-2}{1+\frac{1}{2}\cdot 2}=\frac{-\frac{3}{2}}{2}=-\frac{3}{4}$$

$\varphi=arc\,tg(-\frac{3}{4})$ ugao pod kojim se sijeku date krive u tački $(1;1)$.

(5.) Naći ugao pod kojim se sijeku parabole $y=(x-2)^2$ i $y=-4+6x-x^2$.

Rj. $\varphi=40°36'$

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Izvodi višeg reda

$y = f(x)$ – data f-ja

$y' = f'(x)$ prvi izvod

$y'' = (f'(x))' = f''(x)$ drugi izvod

$y''' = [f''(x)]' = f'''(x)$ treći izvod

$\vdots$

$y^{(n)} = [y^{(n-1)}]' = f^{(n)}(x)$ n-ti izvod f-je $y = f(x)$

1° Nađi y''' f-je $y = xe^x$

Rj. $y = xe^x$

$y' = e^x + xe^x = (x+1)e^x$

$y'' = e^x + (x+1)e^x = (x+2)e^x$

$y''' = e^x + (x+2)e^x = (x+3)e^x$

2° Nađi $y^{(5)}$ f-je $y = 2x^3 + 3x^2 - 4x + 5$

Rj. $y' = 6x^2 + 6x - 4$

$y'' = 12x + 6$

$y''' = 12$

$y^{(4)} = 0$

$y^{(5)} = 0$

3° Nađi y'' f-je $y = \ln \dfrac{x^2+3}{x^2+1}$.

Rj. $$y' = \frac{1}{\frac{x^2+3}{x^2+1}} \cdot \left(\frac{x^2+3}{x^2+1}\right)' = \frac{x^2+1}{x^2+3} \cdot \frac{2x\cdot(x^2+1) - (x^2+3)\cdot 2x}{(x^2+1)^2}$$

$$y' = \frac{2x^3+2x-2x^3-6x}{(x^2+3)(x^2+1)} = \frac{-4x}{(x^2+3)(x^2+1)} = \frac{-4x}{x^4+4x^2+3}$$

$$y'' = \frac{(-4)(x^4+4x^2+3)-(-4x)(4x^3+8x)}{(x^2+3)^2(x^2+1)^2} = \frac{-4x^4-16x^2-12+16x^4+32x^2}{(x^2+3)^2(x^2+1)^2} = \frac{12x^4+16x^2-12}{(x^2+3)^2(x^2+1)^2}$$

$$y'' = \frac{4(3x^4+4x^2-3)}{(x^2+3)^2(x^2+1)^2}$$

4. Nađi y'' f-je $y=(x-1)\,e^{-\frac{1}{x+1}}$

Rj. $y'=\left((x-1)e^{-\frac{1}{x+1}}\right)' = e^{-\frac{1}{x+1}} + (x-1)e^{-\frac{1}{x+1}}\cdot\left(-\frac{1}{x+1}\right)' =$

$= e^{-\frac{1}{x+1}} + (x-1)\cdot\frac{1}{(x+1)^2}\,e^{-\frac{1}{x+1}} = \left(1+\frac{x-1}{(x+1)^2}\right)e^{-\frac{1}{x+1}}$

$\left(-\frac{1}{(x+1)}\right)' = \left[-(x+1)^{-1}\right]' = (x+1)^{-2}$ $\qquad y' = \frac{(x+1)^2+x-1}{(x+1)^2}\,e^{-\frac{1}{x+1}}$

$$y' = \frac{x^2+2x\cancel{+1}+x\cancel{-1}}{(x+1)^2}\,e^{-\frac{1}{x+1}} = \frac{x(x+3)}{(x+1)^2}\,e^{-\frac{1}{x+1}} = \frac{(x^2+3x)\,e^{-\frac{1}{x+1}}}{x^2+2x+1}$$

$$y'' = \left[\frac{x(x+3)\,e^{-\frac{1}{x+1}}}{(x+1)^2}\right]' = \frac{\left[(2x+3)e^{-\frac{1}{x+1}}+(x^2+3x)e^{-\frac{1}{x+1}}\cdot\frac{1}{(x+1)^2}\right]\cdot(x+1)^2-(x^2+3x)e^{-\frac{1}{x+1}}\cdot 2(x+1)}{(x+1)^4}$$

$$y'' = \frac{\left[(2x+3)\overbrace{(x+1)^2}^{x^2+2x+1}+x^2+3x-2\overbrace{(x^2+3x)(x+1)}^{x^3+x^2+3x^2+3x}\right]e^{-\frac{1}{x+1}}}{(x+1)^4}$$

$$y'' = \frac{\cancel{2x^3}+4x^2+2x+3x^2+6x+3+x^2+3x\cancel{-2x^3}-8x^2-6x}{(x+1)^4}\,e^{-\frac{1}{x+1}}$$

$$y'' = \frac{5x+3}{(x+1)^4}\,e^{-\frac{1}{x+1}}$$

5. Nađi y'' f-ja:

a) $y=\frac{x^3}{x^2-2x-8}$ $\qquad$ Rj. $y''=\frac{24x(x^2+4x+16)}{(x^2-3x-8)^3}$

b) $y=\frac{16}{x^2\cdot(x-4)}$ $\qquad$ Rj. $y''=\frac{64(3x^2-16x+24)}{x^4(x-4)^3}$

c) $y=(2x-1)\,e^{-\frac{x}{x-1}}$ $\qquad$ Rj. $y''=\frac{e^{-\frac{x}{x-1}}}{(x-1)^4}$